McGILL UNIVERSITY
FACULTY OF SCIENCE

FINAL EXAMINATION

HONORS ANALYSIS 3, MATHEMATICS 354

Examiner: Professor Jakobson
Associate Examiner: Professor Toth

Date: Friday, December 14, 2012
Time: 14:00 - 17:00

INSTRUCTIONS

Answer all questions. Please give a detailed explanation for each answer.
You may use any result proved in class or in the book, but must state
precisely the statement that you are using.
Non-programmable calculators are permitted.
This is a closed-book exam
Dictionaries are permitted

Dmitry Jakobson

This exam comprises the cover and two pages of questions.
Problem 1 (8 points).

a) (4 points) Let \( f_n : [0, 1] \to \mathbb{R} \) be a sequence of measurable functions on \([0, 1]\).
State measurability properties of \( \liminf_{n \to \infty} f_n(x) \) and \( \limsup_{n \to \infty} f_n(x) \). You don’t need to prove these properties.

b) (4 points) Let \( f_n : [0, 1] \to \mathbb{R} \) be a sequence of measurable functions on \([0, 1]\).
Prove that the set \( \{ x : \lim_{n \to \infty} f_n(x) exists \} \) is Lebesgue measurable.

Problem 2 (8 points).

a) (4 points) State Monotone and Dominated convergence theorems. You don’t need to prove them.

b) (4 points) Find the limit as \( n \to \infty \)

\[
\int_0^n (1 - (x/n))^n e^{x^2/2} \, dx
\]

and justify your answer.

Problem 3 (8 points).

Given a measurable function \( f \in L^1([0,1]) \), define a function \( F_f : \mathbb{R}_+ \to \mathbb{R}_+ \) (the distribution function of \( f \)) by

\[
F_f(t) := \mu\{x : |f(x)| > t\}.
\]

a) (4 points) Prove that

\[
F_f(t) \leq \frac{||f||_1}{t}.
\]

Hint: recall Chebyshev’s inequality.

b) (4 points) Let \( f = g + h \). Prove that \( F_f(t) \leq F_g(t/2) + F_h(t/2) \).
Problem 4 (10 points).

a) (2 points) Define when a subset of a metric space is connected.

b) (2 points) Define when a subset of a metric space is path connected. What is the relationship between connected and path connected? You don’t need to prove anything.

c) (2 points) State the Intermediate Value theorem for connected sets.

d) (4 points) Prove that the set $B$ of $(x, y) \in \mathbb{R}^2$ such that $\{(x, y) : 1 \leq |x|+|y| \leq 2\}$ is connected. Prove that the function $g(x, y) = e^{x+y}$ attains the value 7 on the set $B$. You may use the fact that $2 < 2.7 < e < 2.8 < 3$.

Problem 5 (8 points).

a) (4 points) State Egorov’s theorem.

b) (4 points) Verify the conclusion of Egorov’s theorem for a sequence of functions $\{f_n\} : [0, 1] \to \mathbb{R}$ defined by $f_n(x) = \sin(1/(nx))$, $x > 0$ and $f_n(0) = 0$ for $n = 1, 2, \ldots$.

Problem 6 (8 points).

a) (2 points) Define when a subset of a metric space is closed.

b) (2 points) State the properties of the closed sets under the union and intersection operations.

c) (4 points) Let $X$ be a metric space and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be continuous functions. Show that the set $\{x \in X : f(x) = g(x)\}$ is closed.