McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 354

Examiner: Professor Jakobson          Date: Monday, December 10, 2010
Associate Examiner: Professor Jaksic   Time: 14:00 - 17:00

INSTRUCTIONS

Answer any all questions. Please give a detailed explanation for each answer. You may use any result proved in class or in the book, but must state precisely the statement that you are using.
Non-programmable calculators are permitted.
This is a closed-book exam
Dictionaries are permitted

This exam comprises the cover and two pages of questions.
Problem 1 (6 points).
Let \( f(x), g(x) \in C([0,1]) \). Assume that
\[
\int_0^1 |f(x) - g(x)|^3 dx < 27,
\]
and that
\[
\int_0^1 |f(x)^2 + f(x)g(x) + g(x)^2|^{3/2} dx < 8.
\]
Prove that
\[
\int_0^1 |f(x)^3 - g(x)^3| dx \leq 12.
\]

Problem 2 (8 points).

a) (2 points) Define when a subset of a metric space is connected.

b) (2 points) State the Intermediate Value theorem for metric spaces.

c) (2 points) Prove that the set \( B \) of \( (x,y) \in \mathbb{R}^2 \) such that \( \{ (x,y) : 1 \leq \sqrt{x^2+y^2} \leq 2 \} \) is connected.

d) (2 points) Does the function \( g(x, y) = \cosh(x^2+y^2) \) attain the value 15 on the set \( B \)? You may use the fact that \( 2 < 2.7 < e < 2.8 < 3 \).

Problem 3 (8 points).

a) (2 points) Define when a subset of a metric space is closed.

c) (2 points) State basic properties of closed sets with respect to intersections and unions.

c) (4 points) Let \( F_j(x_1, \ldots, x_n), 1 \leq j \leq m \) be polynomials in \( n \) variables. Prove that the set of solutions of the system of algebraic equations
\[
\begin{cases}
F_1(x_1, \ldots, x_n) = a_1, \\
\quad \ldots \\
F_m(x_1, \ldots, x_n) = a_m.
\end{cases}
\]
is a closed subset of \( \mathbb{R}^n \).
Problem 4 (10 points).
   a) (2 points) Define when a subset of a metric space is dense.
   b) (2 points) Define when a metric space is separable.
   c) (3 points) Is $l_2$ separable? Prove or disprove.
   d) (3 points) Is $l_\infty$ separable? Prove or disprove.

Problem 5 (10 points). Let $X$ be a normed linear space.
   a) (2 points) State equivalent definitions of a continuous linear functional on $X$.
   b) (2 points) Define the norm of a linear functional.
   c) (3 points) Prove that the set of all bounded linear functionals on $X$ themselves form a normed linear space (called the conjugate space $X^*$ of $X$).
   d) (3 points) Describe the conjugate space of $l_2$ (you don’t need to prove that result). Give a formula for the norm of a linear functional on $l_2$, and prove it.

Problem 6 (8 points). Let $X$ denote the space $C[0,1]$ with the $d_\infty$ distance, $||f||_\infty = \max_{x \in [0,1]} |f(x)|$. Prove that the following operators $T$ from $X$ to itself are continuous, and find their norms.
   a) $Tf(x) = f(x^\alpha), \alpha > 0$.
   b) $Tf(x) = \int_0^x f(t)dt$. 