MCGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 354

Examiner: Professor Jakobson
Associate Examiner: Professor Klemes

Date: Monday, December 18, 2006
Time: 14:00 - 17:00

INSTRUCTIONS

Answer any 6 of the following 7 questions. Each question is worth 10 points. Please give a detailed explanation for each answer. You may use any result proved in class or in the book, but must state precisely the statement that you are using.
Non-programmable calculators are permitted.
This is a closed-book exam
Dictionaries are permitted

This exam comprises the cover and two pages of questions.
Problem 1. (10 points) Let $X$ be a metric space with the distance $\rho$, and $Y$ be a metric space with the distance $\sigma$. Let $X \times Y$ be the set of pairs $\{(x, y) : x \in X, y \in Y\}$. Define a distance $d$ on $X \times Y$ by

$$d((x_1, y_1), (x_2, y_2)) = \max\{\rho(x_1, x_2), \sigma(y_1, y_2)\}.$$

Note that this defines, e.g., the $d_\infty$ distance on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

a) (3 points) Prove that $d$ defines a distance on $X \times Y$.

b) (3 points) Define when a metric space is totally bounded.

c) (4 points) Prove that if $(X, \rho)$ and $(Y, \sigma)$ are totally bounded, then so is $(X \times Y, d)$.

Problem 2. (10 points)

a) (3 points) Define when a family of functions is (uniformly) equicontinuous.

b) (3 points) State the Arzela-Ascoli theorem.

c) (4 points) Consider the space $X = C([0,1])$ with the supremum distance,

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|,$$

induced by the norm $||f|| = \sup_{x \in [0,1]} |f(x)|$. Let $Y$ be a bounded subset of $X$. Prove that the set of functions

$$F(x) = \int_0^x t \cdot f(t) \, dt, \quad f \in Y$$

has compact closure.

Problem 3. (10 points)

a) (2 points) Define when a subset of a metric space is compact.

b) (3 points) Give an equivalent definition of compactness in $\mathbb{R}^n$.

c) (5 points) Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, and $f(x) \geq ||x||$ for all $x \in \mathbb{R}^n$. Prove that $f^{-1}([0,1])$ is compact.
Problem 4. (10 points)

a) (2 points) Define when a subset of a metric space is connected.

b) (2 points) State the Intermediate Value theorem for metric spaces.

c) (3 points) Prove that the set $B$ of $(x, y) \in \mathbb{R}^2$ such that $(x - 1)^2 + (y - 2)^2 \leq 5$ is connected.

b) (3 points) Does the function $g(x, y) = \exp(|x| + |y|)$ attain the value 50 on the set $B$? You may use the fact that $2 < 2.7 < e < 2.8 < 3$.

Problem 5. (10 points)

a) (2 points) State the Contraction Mapping theorem.

b) (6 points) Let $Y$ be the set of continuous functions on $[0, 1]$ that take values in $[0, 1]$, with the uniform distance. Prove that $Y$ is complete. Also, prove that the mapping $A$ defined on $Y$ by the formula $[Af](x) = [(f(x))^3 + x + 2]/4$ is a contraction mapping of $Y$ into itself.

c) (2 points) Conclude that there exists a unique function $f : [0, 1] \rightarrow [0, 1]$ satisfying $f(x) = [(f(x))^3 + x + 2]/4$.

Problem 6. (10 points)

a) (4 points) Define when a sequence of functions (from a metric space to another metric space) converges uniformly.

b) (6 points) Consider the sequence of functions $f_n : [0, \pi] \rightarrow \mathbb{R}$ defined by the formula $f_n = \sin \circ \sin \circ \ldots \circ \sin$ (taken $n$ times), i.e. $f_1(x) = \sin x$, $f_2(x) = \sin(\sin x)$, $f_3(x) = \sin(\sin(\sin x))$ etc. Prove that the sequence of functions $\{f_n\}$ converges uniformly to the zero function as $n \rightarrow \infty$. You can use the fact that $0 < \sin x < x$ for $0 < x < \pi$.

Problem 7. (10 points)

a) (4 points) State the Implicit Function theorem.

b) (6 points) Let $x, y, u, v$ be related by

$$xe^{u+v} + 2uv - 1 = 0, \quad ye^{u-v} - \frac{u}{1+v} - 2x = 0.$$

Compute partial derivatives $(\partial u/\partial x), (\partial v/\partial y)$ at the point where $x = 1, y = 2$ and $u = v = 0$. 