Department of Mathematics and Statistics
McGill University

FINAL EXAMINATION
Math354, Honours Analysis 3

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Date & time: Monday December 12, 2005, 14:00-17:00

Instructions:

• This is a closed-book exam.
• You are allowed to use non-programmable calculators.
• Give detailed and complete solutions.
• Attempt all problems.
• Write your answers in the exam booklet.
• The use of a regular dictionary is allowed.

This exam consists of a cover sheet plus one page containing eight questions.
1. [5pt] Let $X$ be a separable metric space and let $Y \subseteq X$. Show that $Y$ is separable as well.

2. [5pt] Let $k : [0, 1] \times [0, 1] \to \mathbb{R}$ be a continuous function and consider the map $K : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ given by

$$(Kf)(x) = \int_0^1 k(x, y)f(y)\,dy.$$  

Show that any sequence $f_n \in C([0, 1], \mathbb{R})$ satisfying $\|f_n\| \leq 1$ has a subsequence $f_{n_j}$ with $Kf_{n_j}$ uniformly convergent.

(Suggestion: Use the Ascoli–Arzelà theorem.)

3. [6pt] Let $X$ and $Y$ be compact metric spaces. Show that $X \times Y$ is compact in two ways:
   a) by using sequential compactness,
   b) by using completeness and total boundedness.

4. [4pt] Let $\{A_n\}$ be a collection of connected subsets of a metric space $X$, such that $A_n \cap A_{n+1} \neq \emptyset$ for all $n$. Show that $\bigcup_n A_n$ is connected.

5. [4pt] Let $X = \{(0,0)\} \cup \{(x, \sin(x) \sin(1/x)) \mid 0 < x \leq 1\} \subset \mathbb{R}^2$. Is $X$ path-connected? Justify your answer!

6. [5pt] Let $L \in BL(\mathbb{R}^n)$ be an invertible map, and let $g \in C^1(\mathbb{R}^n)$ be such that $\|g(x)\| \leq M\|x\|^2$. Show that $f(x) = Lx + g(x)$ is locally invertible near 0.

7. [5pt] Let $f$ be a map of class $C^1$ on a Banach space $X$ such that $f(tx) = tf(x)$ for all real $t$ and all $x \in X$. Show that $f$ is linear, and in fact that $f(x) = Df(0)x$.

8. [6pt] Show that the system

$$\begin{align*}
xy^2 + xzu + yv^2 &= 3 \\
u^3yz + 2xv - u^2v^2 &= 2
\end{align*}$$

has a $C^\infty$ solution $u(x, y, z)$, $v(x, y, z)$ near $(x, y, z) = (1, 1, 1)$, $(u, v) = (1, 1)$. Find $\frac{\partial}{\partial y}v(1, 1, 1)$. 