INSTRUCTIONS

1. All questions carry equal weight.
2. Answer 6 or 7 questions; credit will be given for the best 6 answers.
3. Answer questions in the exam book provided. Start each answer on a new page.
4. This is a closed book exam.
5. Notes and textbooks are not permitted.
6. Non-programmable calculators are permitted.
7. Translation dictionaries (English-French) are permitted.

This exam comprises of the cover page, and 3 pages of 7 questions.
1. (a) Show that \( f(x) = x^3 - 3 \) has exactly one zero in the interval \([1, 2]\).
(b) Starting from this interval with \( x_0 = 1.5 \), use three steps of the bisection method to obtain an approximation \( x_3 \) to this zero.
(c) How many steps would be required to ensure that
\[
|x_n - x^*| \leq 10^{-8},
\]
where \( x^* \) is the zero of \( f(x) \).
(d) The following three iterative methods are proposed to compute \( \sqrt[3]{3} \). Rank them in order, based on the order of convergence in a neighbourhood of the (positive) root.

\[
\begin{align*}
(i) \quad x_{n+1} &= x_n - x_n^3 + 3, \\
(ii) \quad x_{n+1} &= \frac{2x_n}{3} + \frac{1}{x_n^2}, \\
(iii) \quad x_{n+1} &= x_n - \frac{(x_n^6 - 9)}{12x_n^2}.
\end{align*}
\]

2. (a) Let \( f(x) \) be \( n + 1 \) times continuously differentiable on \([a, b]\) and \( x_0, x_1, \ldots, x_n \) be distinct interpolation points in \([a, b]\). Define the fundamental Lagrange polynomials \( l_0(x), l_1(x), \ldots, l_n(x) \) for the interpolation points and show that
\[
p_n(x) = \sum_{j=0}^{n} f(x_j) l_j(x)
\]
interpolates \( f \) at \( x_0, x_1, \ldots, x_n \).
(b) Show that \( p_n(x) \) is the unique interpolating polynomial of degree \( n \).
(c) Suppose that \( n = 3 \)
\[
x_0 = 0, \quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 4,
\]
\[
f(x_0) = 1, \quad f(x_1) = \frac{1}{\sqrt{2}}, \quad f(x_2) = 0, \quad f(x_3) = -1.
\]
Find \( p_3(x) \) and evaluate \( p_3(3) \).
(d) Find a bound for the error in this approximation of \( f(3) \), when \( \max_{x \in [0,3]} |f^{(4)}(x)| \leq 1/6 \), using the error formula
\[
f(x) = p_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i).
\]

3. (a) What is the key difference between Lagrange and Hermite interpolants? What is the difference between a clamped and a natural cubic spline?
(b) A natural cubic spline \( S(x) \) on \([1, 3]\) has the formula
\[
\begin{align*}
S_0(x) &= 4 + (x - 1) - (x - 1)^3, \quad \text{if } 1 \leq x \leq 2 \\
S_1(x) &= a + b(x - 2) + c(x - 2)^2 + d(x - 2)^3, \quad \text{if } 2 \leq x \leq 3.
\end{align*}
\]
Find \( a, b, c, d \).
(c) A cubic Bezier curve \( \mathbf{B}(t) \) has end points \( \mathbf{b}_0 = (0, 0) \) and \( \mathbf{b}_3 = (1, 0) \) and guide points \( \mathbf{b}_1 = (0, 1/2) \) and \( \mathbf{b}_2 = (1, 1/2) \). What is the role of the guide points and what properties does the curve have with respect to the four given vectors? State the formula of the curve \( \mathbf{B}(t) \).
4. (a) Show that 
\[ f''(x_0) = \frac{1}{h^2} \left( f(x_0 + h) - 2f(x_0) + f(x_0 - h) \right) - \frac{h^2}{12} f^{(4)}(\xi), \]
where \( \xi \in [x_0 - h, x_0 + h] \).
(b) Given the data

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.9</th>
<th>1.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-0.0948</td>
<td>0.1048</td>
</tr>
</tbody>
</table>

Use the finite difference formula above to approximate \( f''(1) \).
(c) Suppose \(|f^{(4)}(x)| \leq M\) for all \( x \), and that \( h > 0 \). If we encounter roundoff errors \( \delta_i \) in computing \( f(x_0 + ih) \) for \( i = -1, 0, 1 \) and and \( |\delta_i| < \delta \), find an upper bound on the total error in the approximation to \( f''(x_0) \). Determine the value of \( h \) which minimises this bound, if \( \delta = 1 \times 10^{-16} \) and \( M = 3 \).

5. (a) Define the degree of accuracy (also known as the degree of precision) of a quadrature formula \( I_h(f) \) for approximating the integral 
\[ I(f) = \int_a^b f(x)dx. \]
(b) Find the degree of accuracy \( p \) of the quadrature formula 
\[ I_h(f) = \frac{3}{8}h[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] \]
where \( a = x_0 \), \( b = x_3 \) and \( h = x_{i+1} - x_i \).
(c) Given that \( I(f) = I_h(f) + kh^{p+2}f^{(p+1)}(\xi) \), where \( p \) is the degree of accuracy, find \( k \).
(d) Evaluate \( I_h(f) \) when \( I(f) = \int_0^1 e^{-x^2}dx \).

6. (a) Let \( I_h(f) \) be the Composite Trapezoidal Rule approximation to 
\[ \ln 2 = I(f) = \int_1^2 \frac{1}{x}dx. \]
Evaluate \( I_h(f) \) when \( h = 0.5 \) and when \( h = 0.25 \).
(b) Derive the error bound 
\[ I(f) - I_h(f) = -\frac{(b - a)}{12} h^2 f''(\xi) \]
for some \( \xi \in [a, b] \), for the Composite Trapezoidal rule, from the error bound for the Trapezoidal rule.
(c) Apply one-step of Richardson extrapolation to the approximations in (a), to find a better approximation to \( \ln 2 \).
(d) Given that \( \ln 2 = 0.69314718 \), compute the relative error in the approximations found in (a) and (c).
7. Consider the initial value problem
\[ y' = f(y), \quad 0 \leq t \leq T, \quad y(0) = \alpha. \]

Suppose you approximate the solution \( y(t) \) using the Runge-Kutta method
\[
\begin{align*}
w_0 &= \alpha, \\
w_{i+1} &= w_i + \frac{h}{2} \left( f(w_i) + f(w_i + hf(w_i)) \right), \quad i = 0, \ldots, N
\end{align*}
\]
with time-step \( h > 0 \).

(a) Define the local truncation error \( \tau_{i+1}(h) \) and use it to determine the order of this method.

(b) Consider the case where
\[ f(y) = \lambda y, \quad \lambda < 0, \]
and
\[
\begin{align*}
i. \ & \text{show that} \quad w_{i+1} = (1 + h\lambda + \frac{(h\lambda)^2}{2})w_i. \\
ii. \ & \text{Under what conditions on} \ h \ \text{does} \ \lim_{i \to \infty} w_i = 0? 
\end{align*}
\]