FACULTY OF ENGINEERING

FINAL EXAMINATION

MATH 271

LINEAR ALGEBRA AND PARTIAL DIFFERENTIAL EQUATIONS

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Date: Friday December 14, 2007
Time: 9:00 AM - 12:00 PM

INSTRUCTIONS

1. Please answer all questions in the exam booklets provided.
2. This is a closed book examination. No books, crib sheets or lecture notes permitted.
3. Calculators are neither permitted nor required.
4. Use of a regular and/or translation dictionary is permitted.

This exam comprises the cover page, 2 pages of 6 questions and 1 page of useful information.
1. (11 Marks)

(a) Obtain carefully the nontrivial null space of the operator \( \mathcal{L} = \frac{d^4}{dx^4} - \alpha^4 \), where \( \alpha \) is a real positive constant.

(b) Show that the solution of
\[
\frac{d^4y}{dx^4} = \alpha^4 y; \quad y(0) = 0, \quad y'(0) = 0, \quad y(L) = 0, \quad y'(L) = 0
\]
is non-trivial provided \( \cos \alpha L \cosh \alpha L = 1 \).

2. (12 Marks) The circuit below is governed by the following system of differential equations

\[
L_1 \ddot{Q}_1 + M \ddot{Q}_2 + \frac{Q_1}{C_1} = 0
\]
\[
M \ddot{Q}_1 + L_2 \ddot{Q}_2 + \frac{Q_2}{C_2} = 0.
\]

If \( L_1 = L_2 = 5, \quad C_1 = C_2 = \frac{1}{8}, \quad M = 3, \quad Q_1(0) = 0, \quad \dot{Q}_1(0) = 0, \quad Q_2(0) = 4, \quad \dot{Q}_2(0) = 0 \), solve for \( Q_1(t) \) and \( Q_2(t) \).

3. (18 Marks)

(a) Obtain the general solution of Laplace’s equation in polar coordinates with the requirement that the solution be periodic of period \( 2\pi \), i.e., solve
\[
\nabla^2 \psi(r, \theta) = 0; \quad \psi(r, \theta + 2\pi) = \psi(r, \theta), \quad \psi_{\theta}(r, \theta + 2\pi) = \psi_{\theta}(r, \theta).
\]

(b) A very long cylinder of radius \( \alpha \) is immersed in a uniform flow parallel to the \( x \)-axis with speed \( V_0 \). By having the center of the cylinder at the origin find the effect of the cylinder on the velocity potential and velocity. Assume that the cylinder is perpendicular to the flow.
4. (22 Marks) Solve and interpret physically the following diffusion equation boundary value problems:

(a) \( \psi_t - \psi_{xx} = h(x, t); \; 0 < x < \pi, \; t > 0 \)

(i) \( \psi_x(0, t) = F(t) \)  
(ii) \( \psi_x(\pi, t) = G(t) \)  
(iii) \( \psi(x, 0) = f(x) \).

(b) \( \frac{1}{\alpha^2} \frac{\partial \psi}{\partial t}(r, \varphi, \theta, t) = \nabla^2 \psi(r, \varphi, \theta, t); \; 0 \leq r < a, \; 0 \leq \varphi \leq \pi, \; 0 \leq \theta < 2\pi, \; t > 0 \)

(i) \( \psi(a, \varphi, \theta, t) = 0 \)  
(ii) \( \psi(r, \varphi, \theta, 0) = 0 \).

5. (23 Marks) Solve the following Poisson’s equation boundary value problems and interpret physically:

(a) \( \nabla^2 \psi(x, y) = -\sin 3y; \; 0 < x < \pi, \; 0 < y < \pi. \)

(i) \( \psi(0, y) = 0 \),  
(ii) \( \psi(\pi, y) = 0 \),  
(iii) \( \psi(x, 0) = 1 \),  
(iv) \( \psi(x, \pi) = 0 \).

(b) \( \nabla^2 \psi(r, \theta) = -2; \; 0 \leq r < 4, \; 0 \leq \theta < 2\pi. \)

(i) \( \psi(4, \theta) = 0. \)

6. (15 Marks) You may assume that the general solution of Laplace’s equation in spherical coordinates, i.e., \( \nabla^2 \psi(r, \varphi) = 0, 0 \leq \varphi \leq \pi, \psi \) finite at \( \varphi = 0 \) and \( \varphi = \pi \), is given by

\[
\psi(r, \varphi) = \sum_{n=0}^{\infty} \left[ A_n r^n + \frac{B_n}{r^{n+1}} \right] P_n(\cos \varphi),
\]

where \( P_n \) are the Legendre polynomials of order \( n \).

Solve \( \nabla^2 \psi(r, \varphi) = 0, \; 0 \leq \varphi < \pi/2, \; 0 \leq r < \alpha, \)

(i) \( \psi(r, \pi/2) = 0, \)  
(ii) \( \psi(\alpha, \varphi) = f(\cos \varphi) \) And as a special case \( f(\cos \varphi) = \cos^3 \varphi. \)

Leave your answer in SIMPLEST terms. Interpret physically.
USEFUL INFORMATION

1. \( \nabla^2 \psi(r, \theta) = \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial \psi}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \).

2. \( \nabla \psi(r, \theta) = \frac{\partial \psi}{\partial r} \hat{u}_r + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \hat{u}_\theta. \)

3. Laplacian in spherical coordinates
\[
\nabla^2 \psi(r, \varphi, \theta) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \varphi} \frac{\partial}{\partial \varphi} \left( \sin \varphi \frac{\partial \psi}{\partial \varphi} \right) + \frac{1}{r^2 \sin^2 \varphi} \frac{\partial^2 \psi}{\partial \theta^2}. \]

Note: \( \varphi \) is the angle that the position vector makes with the z-axis.

4. \( \int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2 \delta_{nm}}{2n + 1} \), where \( P_n(x) \) and \( P_m(x) \) are Legendre polynomials.

\( P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2} (3x^2 - 1), P_3(x) = \frac{1}{2} (5x^3 - 3x). \)

Good Luck!