McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 251

HONOURS ALGEBRA 2

Examiner: Professor H. Darmon
Associate Examiner: Professor O. Kharlampovich

Date: Tuesday April 18, 2006
Time: 2:00 PM - 5:00 PM

INSTRUCTIONS

1. Please answer all questions in the exam booklets provide.

2. This is a closed book exam. No notes or texts allowed.

3. Calculators are not permitted.

4. Use of a regular or translation dictionary is not permitted.

This exam comprises the cover page, and 3 pages of 7 questions.
1. Let $T : V \rightarrow W$ be a surjective linear transformation of vector spaces over a field $F$, and suppose that $W$ is finite-dimensional.
   
   (a) Show that there is a subspace $U \subset V$ with the property that the restriction $T|_U : U \rightarrow W$ is an isomorphism between $U$ and $W$.
   
   (b) Show that $V = U \oplus \ker(T)$.
   
   (c) Prove or disprove: there is only one subspace $U$ of $V$ satisfying (b).

2. Let $V$ be a finite-dimensional real vector space and let $T : V \rightarrow V$ be a linear transformation.
   
   (a) Show that if $\dim(V)$ is odd, then $T$ has an eigenvector.
   
   (b) Give an example to show that this is not true in general when $\dim(V)$ is even.

3. Let $T : V_1 \rightarrow V_2$ be a linear transformation between finite-dimensional vector spaces, and let $T^* : V_2^* \rightarrow V_1^*$ denote the resulting linear transformation on the dual spaces. (Recall from the class that $T^*$ is defined by $T^*(\ell) = \ell \circ T$, for all $\ell \in V_2^*$.)
   
   (a) Show that if $T$ is surjective then $T^*$ is injective.
   
   (b) Show that if $T$ is injective then $T^*$ is surjective.
4. Let $V$ be a vector space of dimension $n$ over a field $F$, and let $T : V \to V$ be a linear transformation whose minimal polynomial has degree $n$ and is irreducible over $F$.

(a) Show that the subring of $\mathcal{L}(V, V)$ defined by

$$K = \{g(T), \quad \text{with } g \in F[x]\}$$

is a field which contains $F$. What is its dimension as a vector space over $F$?

(b) Show that the set $V$ becomes a vector space over $K$ with the scalar multiplication defined by

$$(f(T), v) \mapsto f(T)(v).$$

What is the dimension of $V$ over $K$?

(c) Let $U : V \to V$ be a linear transformation of vector spaces over $F$ which commutes with $T$. Show that $U$ is also linear with respect to the scalar multiplication by $K$, i.e., it is a linear transformation on $V$ viewed as a vector space over $K$.

(d) Use (c) to show that any linear transformation $U$ that commutes with $T$ can be expressed as a polynomial in $T$.

5. Let $V$ be an inner product space over $\mathbb{R}$ and let $T : V \to V$ be a self-adjoint transformation on $V$.

(a) State (without proof) what the spectral theorem tells us about $T$.

(b) Let $U \subset V$ be a subspace of $V$ which is stable under $T$ (i.e., $T(U) \subset U$.) Show that $U$ has a complementary $T$-stable subspace, i.e., a subspace $W \subset V$ such that $T(W) \subset W$ and $V = U \oplus W$.

(c) Give an example to show that the statement of (b) can be false when $T$ is not self-adjoint.
6. Find the linear function \( f(x) = ax + b \) which minimizes the quantity
\[
\int_0^\pi (f(t) - \sin(t))^2 dt
\]

7. True or false? (You do not need to justify your answer.)
(a) If \( T : V \to W \) is a linear transformation, and \( V \) is a finite-dimensional vector space, then
\[
\dim(\ker(T)) + \dim(\text{Im}(T)) = \dim(V).
\]
(b) If \( T : V \to V \) is a linear transformation, and \( V \) is a finite-dimensional vector space, then
\[
\ker(T) \oplus \text{Im}(T) = V.
\]
(c) Every invertible linear transformation on a finite-dimensional complex vector space has a square root.
(d) The characteristic and minimal polynomials of a linear transformation acting on a finite-dimensional complex vector space have exactly the same roots.
(e) A linear transformation is diagonalisable if its characteristic polynomial factors into distinct linear factors.
(f) A linear transformation is diagonalisable if and only if its minimal polynomial factors into distinct linear factors.