McGILL UNIVERSITY
FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 223-001 AND 002

LINEAR ALGEBRA

Examiner: Professor Kharlampovich Date: Thurs. December 15, 2005
Associate Examiner: Professor Loveys Time: 9:00 am - 12:00 pm

INSTRUCTIONS

(a) Answer questions in the exam booklets provided.
(b) All questions carry equal weight.
(c) This is a closed book exam. No computers, notes or text books are permitted.
(d) Simple pocket calculators that have no scientific functions are permitted only.
(e) Use of a regular and or translation dictionary is not permitted.
(f) This exam comprises of the cover page, 2 pages of 8 questions.

(9) EXAM IS PRINTED ONLINE
Solve all problems. Faculty calculators are allowed. Books, notes are not allowed.

1. Find a basis for each of the row space, the column space, and the null space of the following matrix with entries in $\mathbb{C}$. What is its rank?

$$
\begin{pmatrix}
1 & 1 + i & 0 & 3 & 0 \\
2 i & -2 + 2 i & 1 & 2 + 5 i & -i \\
1 + i & 2 i & -2 i & 1 - i & -2 \\
3 & 3 + 3 i & 1 - i & 10 - 3 i & -1 - i
\end{pmatrix}
$$

2. Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix} \right\}$ be a subspace of $\mathbb{R}^4$. Find an orthonormal basis for each of $W$ and $W^\perp$. Also find the projections $\text{proj}_W \vec{v}$ and $\text{perp}_W \vec{v}$ of $\vec{v}$ onto $W$ and onto $W^\perp$, where $\vec{v} = \begin{pmatrix} 0 \\ 4 \\ 2 \\ 2 \end{pmatrix}$.

3. Let $P_3(t)$ be the vector space of polynomials over the reals with degree at most three. Let $T : P_3(t) \rightarrow P_3(t)$ be defined by

$$
Tf(t) = t^2 f''(t) - 3tf'(t) + 3f(t)
$$

(a) Find the matrix $[T]_B$ of $T$ with respect to the standard ordered basis $B = \{1, t, t^2, t^3\}$ of $P_3(t)$. Do the same (i.e., find $[T]_C$) for the nonstandard ordered basis $C = \{1, 1 + t, 1 + t + t^2, 1 + t + t^2 + t^3\}$.

(b) Find a basis for the kernel of $T$ and for the image (range) of $T$.

4. Suppose that $A$ and $B$ are similar matrices, and $\lambda$ is an eigenvalue of $A$. Show that $\lambda$ is also an eigenvalue of $B$, and that the dimensions of the corresponding eigenspaces are the same for $A$ and $B$.

5. Let $A = \begin{pmatrix} 8 & 18 & 8 \\ -4 & -9 & -4 \\ 1 & 2 & 1 \end{pmatrix}$. Find (explicitly) $A^{26}$. Find $P$ such that $P^{-1}AP$ is diagonal.

6. Let $V$ be the vector space of all continuous real-valued functions on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. For $f, g \in V$, define $<f, g> = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x f(x) g(x) dx$. Verify that this gives an inner product on $V$
7. Let \( A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} \) have determinant \( 2 + i \), and \( B = \begin{pmatrix} k & \ell & m \\ d & e & f \\ g & h & j \end{pmatrix} \) have determinant \( 4 - i \). Find the determinant of \( \begin{pmatrix} a + 2ik & d & 3g \\ b + 2i\ell & e & 3h \\ c + 2im & f & 3j \end{pmatrix} \).

8. For each of the following quadratic forms \( Q(x_1, x_2, x_3) \), find an orthogonal substitution that diagonalizes \( Q \). Identify the shape of the graphs of \( Q(x_1, x_2) = 1 \), \( Q(x_1, x_2, x_3) = 1 \).

(a) \( Q(x_1, x_2) = x_1^2 - 4x_1x_2 + x_2^2 \);
(b) \( Q(x_1, x_2, x_3) = 3x_1^2 + 4x_1x_2 + 2x_2^2 - 4x_1x_3 + 4x_3^2 \).