McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 223
LINEAR ALGEBRA

Examiner: Professor Olga Kharlampovich
Associate Examiner: Professor Jim Loveys

Date: Tuesday April 26, 2005
Time: 9:00AM - 12:00PM

INSTRUCTIONS

1. Please answer all questions in exam booklets provided.

2. This is a closed book exam.

3. Simple Pocket Calculators that have no scientific functions are permitted only.

4. Regular or Translation dictionaries are not permitted.

This exam comprises the cover page, and 2 pages of 8 questions.
1. Let \( A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \).

(a) Find all eigenvalues and corresponding eigenvectors.

(b) Find a nonsingular matrix \( P \) such that \( D = P^{-1}AP \) is diagonal.

(c) Find a matrix \( B \) such that \( B^2 = A \).

(d) Find \( f(A) \), where \( f(t) = t^4 - 3t^3 - 6t^2 + 7t + 3 \).

2. Let

\[
A = \begin{bmatrix} 1 & 6 & a \\ -5 & -6a & -25 \\ a & 30 & 25 \end{bmatrix}.
\]

Find the rank of \( A \) for all different values of \( a \).

3. Find the Fourier coefficient \( c \) and the projection \( cw \) of \( v = (3 + 4i, 2 - 3i) \) along \( w = (5 + i, 2i) \) in \( \mathbb{C}^2 \).

4. Prove that if \( \{u_1, \ldots, u_r\} \) is an orthogonal set of vectors, then

\[
||u_1 + \cdots + u_r||^2 = ||u_1||^2 + \cdots + ||u_r||^2.
\]

5. Let \( V = C[-\pi, \pi] \) be the vector space of continuous functions on the interval \([ -\pi, \pi ]\) with inner product defined by \( \langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)dt \). The following is an orthogonal set in \( V \)

\[
\{1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots \}.
\]

Find the Fourier coefficients of \( f(x) = x \), namely the numbers \( a_0, b_k, c_k \) (note that \( b_k \) and \( c_k \) may depend on \( k \)) such that

\[
f(x) = a_01 + b_1 \sin x + c_1 \cos x + b_2 \sin 2x + c_2 \cos 2x + \ldots.
\]

6. Let \( F(x, y) = (3x + 4y, 2x - 5y) \). Find the matrix of \( F \) in the standard basis and also in the basis \( S = \{u_1, u_2\} \), where \( u_1 = (1, 2) \), \( u_2 = (2, 3) \).
7. Determine whether the given set $S$ is a subspace of the vector space $\mathbb{R}^{4\times 4}$ of $4 \times 4$ matrices. If $S$ is a subspace of $V$ compute the dimension of $S$. Explain your answer.

(a) $S$ is the set of $4 \times 4$ matrices whose entries are all integers;
(b) $S$ is the set of $4 \times 4$ matrices whose entries are all greater than or equal to 0;
(c) $S$ is the set of $4 \times 4$ matrices with trace 0;
(d) $S$ is the set of all upper triangular $4 \times 4$ matrices;

(e) $S$ is the set of $4 \times 4$ matrices such that the vector $\begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}$ is in the null space of $A$;

(f) $S$ is the set of $4 \times 4$ matrices with non-zero determinant.

8. Let $A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & -2 & 3 & -1 \end{bmatrix}$
Find orthonormal bases of:

(a) the null space of $A$,
(b) the row space of $A$,
(c) the image of the linear mapping given by $A$. 
