1. (a) Factor the polynomial $p(z) = z^3 + 1 + i$ as a product of linear factors, using polar form for the roots. Sketch a plot of the roots in the complex plane.

(b) Find the Principal Value of $\log(-1 - i\sqrt{3})$.

(c) Determine whether or not
\[
\lim_{z \to 0} \frac{\overline{z}}{z}
\]
exists and prove your assertion.

2. (a) Define the term Isolated Singularity of a function $f(z)$, and the term $\text{Res}(f(z), a)$ where $z = a$ is an isolated singularity of $f(z)$. Then find the residues of
\[
\frac{z^{1/2}}{z^3 - 4z^2 + 4z}
\]
at all isolated singularities, using the principal value of $z^{1/2}$.

(b) $\frac{z^2}{1 - e^z}$ at all isolated singularities.

3. Let the function $f$ be defined by
\[
f(t) = \frac{\sin 2t}{t}, \quad -\infty < t < \infty, \quad t \neq 0,
\]
and set $f(0) = 2$. Evaluate its Fourier Transform $F(\omega)$ for $-\infty < \omega < \infty$ (except at jump discontinuities of $F$), using contour integration. Include the detailed analysis of each part of your contour.

4. Given the complex function
\[
F(s) = \frac{e^{-3s}}{s^2 + s + 1}.
\]
(a) Find its Inverse Laplace Transform $f(t)$, $-\infty < t < \infty$, using contour integration. Include diagrams of your contours for the cases $t > 3$ and $t < 3$.

(b) Write down the integral for $F(s)$ in terms of $f(t)$ (the Laplace Transform). Determine the set of all complex numbers $s$ for which this integral converges absolutely, and sketch the set in the $s$-plane.

5. For the complex function $f(z) = z^{-2}(z - 1)^{-1}(z + 3)^{-1}$,

(a) State the three domains in which a Laurent Series in powers of $z$ is available. Is the series a Taylor series in any of these three cases?

(b) Define Principal Part of a function $f$ near an isolated singularity $z = a$. Determine the Principal Part of the given $f$ near $z = 0$.

(c) Compute the Laurent Series referred to in (a) for the domain containing the point $\sqrt{2} + i\sqrt{2}$.

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6. (a) For \( f(t) = 1 + 3^t, \ t \geq 0 \), find the Z transform \( F(z) \). Find all singularities of \( F(z) \).
(b) For \( G(z) = z^{-3}(z - 4)^{-1} \), find the inverse Z transform \( g(nT) \), \( n \geq 0 \).
(c) Write down the series for \( G(z) \) in terms of \( g(nT) \) in part (b) (the Z transform).
Determine the set of all complex numbers \( z \) for which the series converges absolutely and sketch this set in the \( z \)-plane.

7. Let \( f(z) = z^3 + z^2 + 3z + 16 \).
(a) Prove that if \( |z| \geq 10 \) then \( f(z) \neq 0 \). Hint: find a lower bound on \( |f(z)| \).
(b) Let \( \gamma \) be the closed contour shown below, consisting of the line segment from 10i to \( -10i \) plus the semicircle of radius 10 in the right half plane, traversed counterclockwise. Given that the image of \( \gamma \) under the mapping \( z \rightarrow f(z) \) is as shown below, how many zeros of \( f(z) \) are on \( \gamma \) ? inside \( \gamma \) ? Explain.
(c) How many zeros of \( f(z) \) are there in the whole right half plane ? in the left half plane ? on the y axis ? (Assume the results in (a) and (b) are correct).
FACULTY OF ENGINEERING

FINAL EXAMINATION

MATHEMATICS 189-381B

COMPLEX VARIABLES AND TRANSFORMS

Examiner: Professor I. Klemes  Date: Thursday, April 29, 1999
Associate Examiner: Professor D. Sussman  Time: 9:00 A.M. - 12:00 Noon

INSTRUCTIONS

NO CALCULATORS PERMITTED
Show all work and simplify answers.
Answer all 7 questions.
Keep this exam paper.

This exam comprises the cover and 2 pages of questions.