McGILL UNIVERSITY
FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-355B

ANALYSIS IV

Examiner: Professor J.R. Choksi
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Date: Friday, April 24, 1998
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

CALCULATORS NOT PERMITTED.
Attempt any six (6) questions.
All questions carry equal marks.

This exam comprises the cover and 2 pages of questions.
1. For any subset $E \subseteq \mathbb{R}$ and $a \in \mathbb{R}$, let $E + a = \{x + a : x \in E\}$. Let $m^*$ denote Lebesgue outer measure, and $m$ Lebesgue measure. Prove that

(a) $E + a$ is Borel if and only if $E$ is Borel.

(b) $m^*(E + a) = m^*(E)$, all $E \subseteq \mathbb{R}$, $a \in \mathbb{R}$.

(c) $E + a$ is Lebesgue measurable if and only if $E$ is Lebesgue measurable, and then $m(E + a) = m(E)$.

2. Let $(X, S, \mu)$ be a measure space and $\{f_n\} n \in \mathbb{N}$, a sequence of non-negative measurable functions on $(X, S, \mu)$. State the monotone convergence theorem and Fatou's lemma for the sequence $\{f_n\}$. Assuming the monotone convergence theorem, prove Fatou's lemma. Using Fatou's lemma prove the following:

If $f$, $f_n$, $n \in \mathbb{N}$ are non-negative integrable functions on $(X, S, \mu)$ such that $f_n \to f$ a.e. and \[ \int_X f_n d\mu \to \int_X f d\mu, \] prove that for all $E \in S$, \[ \int_E f_n d\mu \to \int_E f d\mu. \]

3. Compute $\lim_{n \to \infty} \int_0^1 \frac{1 + nx^2}{(1 + x^2)^n} dx$, justifying any interchanges in orders of limits.

4. Let $(X, \mathcal{A}, \mu)$, $(Y, B, \nu)$ be finite measure spaces, $\lambda$ the product measure on $\mathcal{A} \otimes B$. Show that the set of functions of the form

$$\sum_{j=1}^n f_j(x)g_j(y), \quad f_j \in L^2(\mu), \ \ g_j \in L^2(\nu) \quad j = 1, \ldots, n, \ n \in \mathbb{N},$$

is dense in $L^2(\lambda)$.

[Hint: First show that characteristic functions of sets in $\mathcal{A} \otimes B$ can be approximated by such functions.]
5. Let \( \{f_n\}, \{g_n\} \) be two complete orthonormal sequences in \( L^2([0,1], \text{one dimensional Lebesgue measure}) \). Show that the set of functions

\[
\{h_{n,m}(x,y) = f_m(x)g_n(y) : n, m \in \mathbb{N}\}
\]

is a complete orthonormal sequence in \( L^2([0,1] \times [0,1], \text{two-dimensional Lebesgue measure}) \).

[Hint: You may use question 4 above, even if you have not done it!]

6. Let \( \{n_k\} \) be an increasing sequence of positive integers and \( E \) the (Lebesgue measurable) set of all \( x \) in \( (-\pi, \pi) \) such that the sequence \( \{\sin n_k x\} \) converges. Show that \( m(E) = 0 \) where \( m \) is Lebesgue measure.

[Hint: If \( A \) is any measurable subset of \( E \), then \( \int_A \sin n_k x \, dx \to 0 \) as \( k \to \infty \), but

\[
\int_A (\sin n_k x)^2 \, dx \to \frac{1}{2} m(A).
\]

7. Prove or disprove the following. (i.e. if the statement is true, give a proof, if it is false give a counterexample.)

(a) Every non-empty Borel set in \( \mathbb{R} \) is either (a) an at most countable union of non-degenerate intervals or (b) an at most countable union of sets consisting of one point or (c) a finite or countable union of sets of types (a) and (b).

(b) If \( (X, \mathcal{A}, \mu) \), \( (Y, \mathcal{B}, \nu) \) are \( \sigma \)-finite measure spaces, \( \lambda \) is the product measure on \( \mathcal{A} \otimes \mathcal{B} \), \( f(x,y) \) is measurable \( \lambda \) and the repeated integral

\[
\int_X \left\{ \int_Y |f(x,y)| \nu(dy) \right\} \mu(dx) < \infty,
\]

then \( f \) is integrable \( \lambda \).

(c) If \( \{f_n\}, n \in \mathbb{N} \) is an orthonormal sequence in \( L^2(X, S, \mu) \) and for all \( f \in L^2 \), \( ||f||^2 = \sum_{n=1}^{\infty} |(f, f_n)|^2 \) then \( \{f_n\} \) is complete in \( L^2 \).