1. (a) Let \((X, d_1)\) and \((Y, d_2)\) be metric spaces. Define:
   (i) Limit point of a subset \(S\) of \(X\);
   (ii) Cauchy sequence in \(X\);
   (iii) Continuous function \(f : X \to Y\);
   (iv) Equicontinuous family of functions;
   (v) Complete metric space;
   (vi) Compact metric space.

   (b) Define: Differentiable function \(f : \mathbb{R}^n \to \mathbb{R}^m\).

2. Let \((X, d)\) be a metric space, \((f_n)\) a sequence of continuous, real valued functions on \(X\).

   (a) If \(f_n\) converges uniformly on \(X\) to a function \(f\), show that \(f\) is continuous.

   (b) If further \(x_n \to x\) in \(X\), show that \(f_n(x_n) \to f(x)\).

3. State Baire’s Theorem.
   Suppose \((X, d)\) is a complete metric space. Let \(G_1, G_2, \ldots\) be a sequence of open subsets of \(X\). Suppose, in addition, \(G_n\) is dense in \(X\), for each \(n\). Prove that \(\bigcap_{1}^{\infty} G_n\) is also dense in \(X\).

4. Let \((f_n)\) be a uniformly bounded sequence of functions which are Riemann integrable on \([a, b]\). If \(F_n(x) = \int_{a}^{x} f_n(t)dt, \ a \leq x \leq b\), prove that there exists a subsequence \((F_{n_k})\) which converges uniformly on \([a, b]\).

5. (a) Prove that every compact metric space \((X, d)\) is separable.

   (b) Prove that if \((X, d)\) is a compact metric space, then \(C(X, \mathbb{R})\) is a separable metric space.
   \(\text{[Hint: Let} \{x_1, x_2, \ldots\} \text{be a subset of} \ X; \text{if} \ f_n(x) = d(x, x_n) \text{for all} \ x \in X, \text{then} \ \{1, f_1, f_2, \ldots\} \text{generates an algebra in} \ C(X, \mathbb{R}).\]
6. (a) State Tietze’s Extension theorem.

(b) (i) Prove that in every infinite metric space there is an infinite sequence \((x_k)\) such that no limit point of the set \(\{x_1, x_2, \ldots\}\) is an element of the sequence.

(ii) Let \((X, d)\) be a compact metric space and suppose that the bounded closed sets of \(C(X, \mathbb{R})\) are compact; prove that \(X\) consists of a finite number of points.

7. (a) Let \(f\) be a bijection from the open set \(U \subset \mathbb{R}^n\) onto the open set \(V \subset \mathbb{R}^n\).

(i) If \(f\) and \(f^{-1}\) are differentiable on \(U\) and \(V\) respectively, prove that the Jacobian \(J_f(x) \neq 0\) for all \(x \in U\).

(ii) If in (i) we do not assume differentiability of \(f^{-1}\), is the conclusion \(J_f(x) \neq 0\) for all \(x \in U\) still valid?

(iii) Let \(f\) be differentiable in \(U\) and let \(f^{-1}\) satisfy a Lipschitz condition on \(V\). Prove that \(f^{-1}\) is differentiable on \(V\).

(b) Let \(U\) be an open set in \(\mathbb{R}^n\), let \(u_0 \in U\) and let \(f : U \to \mathbb{R}^n\) be continuous on \(U\) and continuously differentiable on \(U \setminus \{u_0\}\). If \(\lim_{x \to u_0} Df(x) = L\), prove that \(f\) is also differentiable at \(u_0\) and \(Df(u_0) = L\).
This exam comprises the cover and 2 pages of questions.