McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 255-001

HONOURS ANALYSIS 2

Examiner: Professor K. Gowrisankaran
Associate Examiner: Professor S. Drury

Date: Wednesday April 12, 2006
Time: 2:00 pm - 5:00 pm

INSTRUCTIONS

(a) Answer questions in the exam booklets provided.
(b) All questions count equally.
(c). This is a closed book exam. No computers, notes or text books are permitted.
(d) Calculators are not permitted.
(e) Use of a regular and or translation dictionary is not permitted.

This exam comprises of the cover page, and 2 pages of 6 questions.
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MATH 255 FINAL EXAMINATION

No Calculators

Answer all questions. All questions count equally.

1. Decide if the following statements are true or false.
   
   (a) \( \sum_{1}^{\infty} n a_n \) converges \( \implies \sum_{1}^{\infty} a_n \) converges
   
   (b) \( f(x) := \sin(1/x) \) is Riemann - Darboux integrable on \([0,1]\)
   
   (c) Suppose \( f \) is continuous on \([1,\infty)\) and \( \int_{1}^{\infty} f \) is finite then \( f(x) \) tends zero as \( n \to \infty \).

2. Justify your conclusions.
   
   (a) \( f_n(x) := nx(1-x^2)^n \) for each \( n, x \in [0,1] \). Prove that \( (f_n) \) converges pointwise but the convergence is not uniform.
   
   (b) Suppose \( (a_n) \) is a bounded sequence of real numbers such that \( \sum_{n=0}^{\infty} a_n \) diverges.

   Show that \( \sum_{0}^{\infty} a_n x^n \) has radius of convergence 1.

3. (a) Find \( \lim_{n \to \infty} \sum_{k=1}^{n} \frac{n}{k^2 + n^2} \) by using the definition of Riemann - Darboux integral of an appropriate continuous function.
   
   (b) Let \( f : [0,1] \to \mathbb{R} \) be continuous. Prove that the Cauchy-Reimann integral
   
   \[ \int_{0}^{1} \frac{f(x)}{\sqrt{1-x^2}} \, dx \]

   is finite.

4. (a) Suppose \( P \) and \( Q \) are polynomial functions of degree \( p \) and \( q \) respectively. Suppose further that whatever be \( k \in \mathbb{N}, \ Q(k) \neq 0 \). Prove that \( \sum_{k=1}^{\infty} (-1)^{k+1} \frac{P(k)}{Q(k)} \) is convergent if and only if \( p < q \).
   
   (b) Show that \( \sum_{1}^{\infty} \frac{1}{n^2} < 2 + \sqrt{2} \).

5. (a) Let \( K_1 \) and \( K_2 \) be two non-void compact subsets of \( \mathbb{R} \) such that \( K_1 \cap K_2 = \emptyset \). Prove that \( \inf \{|x-y| : x \in K_1, y \in K_2\} > 0 \).
(b) Let \((f_n)\) be a sequence of continuous functions on \(\mathbb{R}\) such that \(\forall x \in [0,1], f_n(x) \leq f_{n-1}(x) \quad \forall n \geq 2\). Suppose \(\forall x \in [0,1], f_n(x) \to 0\). Prove that \((f_n)\) converges to 0 uniformly.
[Hint: \(\forall \epsilon > 0,\) prove that \(\{V_m\}\) is an open cover of \([0,1]\) if \(V_m = \{x : f_m(x) < \epsilon\}\)]

6. (a) Define/Explain the following concepts.

(i) \(d\) is a metric on a set \(X\)

(ii) \(x_n \in X\) and \(y \in X, x_n \to y\) in the metric \(d\)

(b) Let \(F\) be a non-void closed subset of a space \(X\) with a metric \(d\) and let 
\(\rho(x,F) = \inf\{d(x,y) : y \in F\}\).

Prove that \(\rho\) is a continuous function on \(X\) such that \(\{x : \rho(x,F) = 0\} = F\).

(c) Let \(F_1 \subset X, F_2 \subset X\) be two disjoint non-void closed subsets of the metric space \((X,d)\). Prove that \(f(x) = \frac{\rho(x,F_1)}{\rho(x,F_1) + \rho(x,F_2)}\) is a continuous function \(X \to [0,1]\).

(d) Use the \(f\) in (c) above and show that there are open sets \(V_j \supset F_j\) such that \(V_1 \cap V_2 = \emptyset\).