1. (a) Define what is meant by a bounded sequence, a convergent sequence, a monotone sequence. Prove that a convergent sequence is bounded. Give an example to show that a bounded sequence need not be convergent.

(b) If \((a_n)\) is a convergent sequence, \((b_n)\) a bounded sequence and \(\lim(a_n) = 0\), show that \((a_nb_n)\) is convergent and \(\lim(a_nb_n) = 0\).

2. (a) If \(S\) is a bounded nonempty set in \(\mathbb{R}\), show that the set \(S' = \{x : -x \in S\}\) is bounded, and determine its bounds in terms of the bounds of \(S\).

(b) Let \(A\) and \(B\) be bounded nonempty subsets of \(\mathbb{R}\), and let \(A + B = \{a + b : a \in A, \ b \in B\}\). Prove that \(\sup(A + B) = \sup A + \sup B\).

3. (a) If \(a_n = \frac{1}{n + 1} + \ldots + \frac{1}{2n}\), for all \(n \in \mathbb{N}\), show that \(\frac{1}{2} \leq a_n \leq 1\) for all \(n\); show further that \((a_n)\) converges.

(b) If \(a_n = \sqrt{n + 1} - \sqrt{n}\) for all \(n \in \mathbb{N}\), show that \((a_n)\) and \((\sqrt{n}a_n)\) both converge.

4. (a) Let \(f : [a, b] \to \mathbb{R}\) be continuous. Prove that \(f\) attains an absolute maximum on \([a, b]\).

(b) Suppose that \(f : (a, b) \to \mathbb{R}\) is continuous and let

\[
\lim_{x \to a^+} f(x) = \lim_{x \to b^-} f(x) = 0.
\]

If \(f \left(\frac{a + b}{2}\right) > 0\) prove that \(f\) attains an absolute maximum at some point \(c \in (a, b)\).

5. Define what is meant by the statement that a function \(f\) is \textit{uniformly continuous} on an interval. \textit{State} a theorem on the uniform continuity of a continuous function on a bounded closed interval. Show that a function \(f\) can be uniformly continuous on every bounded interval and yet not be uniformly continuous on \(\mathbb{R}\). Show however that if in addition \(\lim_{x \to +\infty} f(x)\), \(\lim_{x \to -\infty} f(x)\) both exist and are finite, then \(f\) \textit{is} uniformly continuous on \(\mathbb{R}\).

6. State the Mean-Value Theorem.

Let \(f\) be a differentiable function in \(\mathbb{R}\) with \(|f'(x)| \leq \frac{1}{2}\) for all \(x \in \mathbb{R}\). Prove that for any \(x, y \in \mathbb{R}\)

\[
|f(x) - f(y)| \leq \frac{1}{2}|x - y|.
\]
FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-242A

REAL ANALYSIS I

Examiner: Professor J.R. Choksi
Associate Examiner: Professor S.W. Drury

Date: Tuesday, December 10, 1996
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

No Calculators, Notes or Books Permitted
All questions carry equal marks

This exam comprises the cover and 1 page of questions.